Math 131 B, Lecture 1
Real Analysis

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1. *10pts.*

Let (S, d_S) and (T, d_T) be two metric spaces, each having more than one point. Their Cartesian product is the set $S \times T = \{(s,t) : s \in S, t \in T\}$. Below are two proposed metrics for $S \times T$. Which is a valid metric? Justify your answer.

$$d_1((s_1, t_1), (s_2, t_2)) = d_S(s_1, s_2) + d_T(t_1, t_2)$$
$$d_2((s_1, t_1), (s_2, t_2)) = d_S(s_1, s_2) \cdot d_T(t_1, t_2)$$

Solution: We see that d_2 is not a valid metric: consider (s,t) and (s,t'), where the second coordinate differs but not the first. Then we have $d_2((s,t),(s,t')) = d_S(s,s) \cdot d_T(t,t') = 0 \cdot d_T(t,t') = 0$. But $(s,t) \neq (s,t')$, so this violates positivity.

However, it's straightforward to check that d_1 is valid metric.

Problem 2.

(a) [5pts.] Let (M, d) be a metric space, and $S \subseteq M$. Give a definition of a limit point of S.

Solution: We say that $p \in M$ is a limit point of S if every ball $B(p; r)$ contains a point of S other than p.

(b) [5pts.] We say that $S \subset M$ is a dense subset of M if every open set in M contains a point of S. Prove that if S is dense in M, $\overline{S} = M$.

Solution: It suffices to show that any point p of $M - S$ is a limit point of S. Let $B(p; r)$ be any ball around p, then since $B(p; r)$ is open in M and S is dense in M, $B(p; r)$ must contain a point of S (other than p, since $p \notin S$). Ergo p is a limit point of S, and $\overline{S} = M$.

Problem 3.

(a) [5pts.] Give a definition of a compact set.

Solution: We say that $S \subset M$ is compact if for every open covering $\mathcal{F} = \{A_\alpha\}$ of S there is a finite subcover $A_1, \cdots, A_n$ such that $S \subset \bigcup_{i=1}^n A_i$.

[Editorial note: A very common mistake is stating this definition with some form of "there is an open covering with a finite subcover" or trying to use this formulation in a proof. For a space to be compact, *every* open covering has to have a finite subcover.]

(b) [5pts.] Let S be compact and $X \subset S$ be closed. Prove that X is also compact.

Solution: Suppose that X has a cover $\mathcal{F} = \{A_\alpha\}$ of open sets in S. Then $\mathcal{F} \cup (S - X)$ is an open cover of S. (Note that $S - X$ is open in S since X is closed. Ergo some finite subcollection $S - X, A_1, \cdots, A_n$ covers S. But this implies that $A_1, \cdots, A_n$ covers X, since $S - X$ contains no points of X. So a finite subcollection of $\mathcal{F}$ covers X. Since $\mathcal{F}$ was arbitrary, X is compact.

Problem 4.

(a) [5pts.] Let $\{\mathbf{x}^k\}$ be a sequence in $\mathbb{R}^n$, where each $\mathbf{x}^k = (x_1^k, \cdots, x_n^k)$. Prove that $\{\mathbf{x}^k\}$ converges in $\mathbb{R}^n$ if and only if each sequence $\{x_i^k\}$ converges in $\mathbb{R}$.

Solution: Notice that for any points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $|x_i - y_i| \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sqrt{n} \max\{|x_i - y_i| : 1 \leq i \leq n\}$, where the first inequality is valid for any i. We can use the first inequality to show that if $\{\mathbf{x}^k\}$ converges, each $\{x_i^k\}$ converges, and the second to show that if each $\{x_i^k\}$ converges

(b) [5pts.] Use part (a) to give a short proof that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. (Hint: Pick a subsequence that converges in the first coordinate, then look at the second coordinate...)

Solution: Let $\{\mathbf{x}^k\}$ be a sequence in $\mathbb{R}^n$. Consider the sequence of real numbers $\{x_1^k\}$. It has a convergent subsequence $\{x_1^{s_k}\}$, by Bolzano-Weierstrass for $\mathbb{R}$. Now consider the sequence of real numbers $\{x_2^{s_k}\}$ (that is, the second coordinates of the subsequence we chose). This sequence has a subsequence $\{x_2^{t_k}\}$ which converges; moreover, since subsequences of convergent sequences converge to the same limit, $\{x_1^{t_k}\}$ is still a convergent subsequence of $\{x_1^k\}$. Repeat for each coordinate, passing to a subsequence at each step, until we have a subsequence $\{\mathbf{x}^{w_k}\}$ such that in each coordinate, the sequence $\{x_i^{w_k}\}$ converges. By part (a), $\{\mathbf{x}^{w_k}\}$ converges in $\mathbb{R}^n$.

Problem 5.

(a) [5pts.] State the Cantor Intersection Theorem.

Solution: Let $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq$ be a nested sequence of nonempty closed subsets in $\mathbb{R}^n$ such that Q_1 is bounded. Then $\bigcap_{i=1}^n Q_i$ is nonempty.

(b) [5pts.] The *Cantor set* is a subset of the real line constructed as follows: Let $Q_1 = [0, 1]$ and Q_2 be obtained from Q_1 by removing the middle third of the interval, i.e. $Q_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then to obtain Q_3, we cut out the middle thirds of the two remaining intervals, so that $Q_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We continue to construct Q_n by removing the middle third of each interval in Q_{n-1}. The Cantor set is $\bigcap_{i=1}^{\infty} Q_i$. The first three stages are drawn below.

Prove that the Cantor set contains infinitely many points. [Hint: The quickest way to do this is to prove that the Cantor set has more than 2^n points for any n.]

Solution: Notice that $Q_1 \supseteq Q_2 \supseteq \cdots$, each Q_n is closed, and Q_1 is bounded. Ergo by the Cantor Intersection Theorem, the Cantor set is definitely nonempty. However, notice that Q_n consists of 2^n disjoint intervals. Each of these intervals is at the beginning of its own chain of nested closed subsets, so the ultimate intersection $\cap_{i=n}^{\infty} Q_i$ must contain at least one point from each interval, so at least 2^n points. Therefore the Cantor set contains more that 2^n points for any n, and we conclude that it is infinite.